Symmetry properties and exact patterns in birefringent optical fibers

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A pair of nonlinear Schrödinger equations, describing the propagation of waves in birefringent optical fibers, is studied by means of a Lie group technique. The symmetry algebra and the symmetry group associated with the equations are exploited to provide exact configurations. These are the soliton profile, which corresponds to a linear combination of the coordinate translations and the constant change of phase, a solution expressed in terms of the sinus elliptic function, a solution related to the Galilean boost, and other solutions which may be used as a guide for the creation of different experimental patterns. Among them, of special interest is a configuration involving the loss coefficient of the fiber, whose "mass density" is time independent and behaves as a screened Coulomb potential in the space variable.

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I. INTRODUCTION

The propagation of optical pulses in nonlinear birefringent fibers is described by the pair of nonlinear Schrödinger equations [1]

$$\Delta_1 = i\,u_x + u_{tt} + kv + (\alpha|u|^2 + \beta|v|^2)u = 0, \quad (1.1a)$$

$$\Delta_2 = i\,v_x + v_{tt} + ku + (\alpha|v|^2 + \beta|u|^2)v = 0, \quad (1.1b)$$

where $u = u(x,t)$ and $v = v(x,t)$ are the circularly polarized components of the optical field, x and t denote the (normalized) longitudinal coordinate of the fiber and the time variable, respectively, k is the birefringence parameter, and the coefficients α and β are responsible for the nonlinear properties of the fiber [2–4]. Performing the change of variables $x \longrightarrow \alpha x$, $t \longrightarrow \mu t$, with $\mu^2 = \pm\frac{1}{2}\alpha$, Eqs. (1.1) take the form

$$i\,u_x \pm \frac{1}{2}u_{tt} + k'v + (|u|^2 + \sigma|v|^2)u = 0, \quad (1.2a)$$

$$i\,v_x \pm \frac{1}{2}v_{tt} + k'u + (|v|^2 + \sigma|u|^2)v = 0, \quad (1.2b)$$

where $k' = k/\alpha$, + (−) holds in the anomalous (normal) dispersion regime and $\sigma = \frac{\beta}{\alpha} = \frac{1+B}{1-B}$, B being the third-order susceptibility coefficient [2–4]. For $\alpha = \beta$ $(\sigma = 1)$ and $k = 0$, the system (1.1) has an infinite set of constants of motion and may be solved by the inverse scattering method [5].

In general, i.e., for $\alpha \neq \beta$ and $k \neq 0$, the system (1.1) is not integrable by inverse scattering. In this case, Eqs. (1.1) possess three constants of motion only [1]. On the other hand, it is well known that a powerful tool for handling both integrable and nonintegrable differential equations is represented by the so-called symmetry approach [6]. This method, which is based on the Lie group theory, consists essentially in looking for symmetry transformations that reduce the equations under consideration to certain ordinary differential equations; each of them comes from an invariant quantity associated with a given symmetry allowed by the system. Following this idea, in this work we apply the symmetry approach to Eqs. (1.1). We display examples of exact solutions in both cases $\alpha = \beta$ and $\alpha \neq \beta$. In this regard, we observe that the birefringent parameter k involved in Eqs. (1.1) is real. However, the symmetry algebra found for $\alpha = \beta$ does not depend on k. This fact suggested that we study the system also for imaginary values of k. In such a situation, at least when $u = v$, k can be interpreted as the loss coefficient of the fiber [7]: We have obtained an interesting exact solution for $u \neq v$ as well. This solution is derived from the Galilean boost. Another important aspect of the symmetry reduction technique is the determination of the infinitesimal operator, which is responsible for the soliton profile and the periodic configuration.

In Sec. II we outline the method of symmetry reduction and obtain the symmetry algebra and the corresponding symmetry group related to Eqs. (1.1). Section III contains examples of specific exact solutions, while in Sec. IV some concluding remarks are reported.

II. METHOD OF SYMMETRY REDUCTION

A. Symmetry algebra

The method of symmetry reduction (SR) consists of an application of the Lie group theory to reduce Eqs. (1.1) to a system of ordinary differential equations.

A fundamental step of the SR procedure is to obtain the Lie point symmetries [6] of Eqs. (1.1): in other words, the symmetry algebra $\mathcal{L}$ and the corresponding symmetry group $\mathcal{G}$ of the equations under investigation. Then, we can buildup solutions that are invariants under some specific subgroup $\mathcal{G}_0$ of $\mathcal{G}$. The SR can be carried out via the determination of the invariants of $\mathcal{G}_0$. Invariants are furnished by the partial differential equations

$$V_j\, I(x,t,u,v,u^*,v^*) = 0, \quad j = 1,2,...,n, \quad (2.1)$$

where $\{V_j\}$ is a basis of the Lie algebra $\mathcal{L}_0$ of $\mathcal{G}_0$, and n is the number of the independent elements (infinitesimal operators) V_j of $\mathcal{L}_0$. Once the invariants related to $\mathcal{G}_0$ are known, Eqs. (1.1) can be written in terms of them. In such a way, we are led to a set of reduced equations which may yield exact solutions to the original system (1.1). The Lie point symmetries of Eqs. (1.1) can be found by resorting to the standard technique outlined in [6]. Precisely, let us introduce the vector field

$$V = \xi_1\partial_x + \xi_2\partial_t + \xi_3\partial_u + \xi_4\partial_{u^*} + \xi_5\partial_v + \xi_6\partial_{v^*}, \qquad (2.2)$$

where ξ_j ($j = 1, 2, ..., 6$) are functions that depend, in general, on x, t, u, u^*, v, v^*, and $\partial_x = \frac{\partial}{\partial x}$, and so on. A local group of transformations G is a symmetry group for Eqs. (1.1) if, and only if,

$$\mathrm{pr}^{(2)}V[\Delta_j] = 0, \quad \mathrm{pr}^{(2)}V[\Delta_j^*] = 0, \quad j = 1, 2, \qquad (2.3)$$

whenever $\Delta_j = 0$, $\Delta_j^* = 0$ for every generator of G, where $\mathrm{pr}^{(2)}V$ is the second prolongation of V [6].

The conditions (2.3) constitute a set of constraints in the form of partial differential equations, which enable us to obtain the coefficients ξ_j. The calculations have been performed in part by using the symbolic computer language REDUCE [8]. We have achieved the following results.

1. *Case I: $\alpha \neq \beta$*

The symmetry algebra is defined by four elements, namely,

$$\begin{aligned} &V_1 = \partial_x, \quad V_2 = \partial_t, \\ &V_3 = i(u\partial_u - u^*\partial_{u^*} + v\partial_v - v^*\partial_{v^*}), \\ &V_4 = x\partial_t + \frac{i}{2}\,t\,(u\partial_u - u^*\partial_{u^*} + v\partial_v - v^*\partial_{v^*}), \end{aligned} \qquad (2.4)$$

where the nonvanishing commutation relations are

$$[V_1, V_4] = V_2, \quad [V_2, V_4] = \frac{1}{2}V_3. \qquad (2.5)$$

2. *Case II: $\alpha = \beta$*

The symmetry algebra is of the sl*(3,C)* type. It is defined by eight elements: four of them coincide with the previous ones, while the others are given by

$$\begin{aligned} V_5 = x\partial_x + \frac{1}{2}t\partial_t - \frac{1}{2}(u\partial_u + u^*\partial_{u^*} + v\partial_v + v^*\partial_{v^*}) \\ +ikx(v\partial_u - v^*\partial_{u^*} + u\partial_v - u^*\partial_{v^*}), \end{aligned} \qquad (2.6a)$$

$$V_6 = i(v\partial_u - v^*\partial_{u^*} + u\partial_v - u^*\partial_{v^*}), \qquad (2.6b)$$

$$\begin{aligned} V_7 = (v\partial_u + v^*\partial_{u^*} - u\partial_v - u^*\partial_{v^*})\cos 2kx \\ -i(u\partial_u - u^*\partial_{u^*} - v\partial_v + v^*\partial_{v^*})\sin 2kx, \end{aligned} \qquad (2.6c)$$

$$\begin{aligned} V_8 = (v\partial_u + v^*\partial_{u^*} - u\partial_v - u^*\partial_{v^*})\sin 2kx \\ +i(u\partial_u - u^*\partial_{u^*} - v\partial_v + v^*\partial_{v^*})\cos 2kx. \end{aligned} \qquad (2.6d)$$

The nonvanishing commutation relations fulfilled by $V_1, \ldots, V_8$, are (2.5) together with

$$\begin{aligned} &[V_1, V_5] = V_1 + kV_6, \quad [V_1, V_7] = -2kV_8, \\ &[V_1, V_8] = 2kV_7, \quad [V_2, V_5] = \frac{1}{2}V_2, \quad [V_5, V_4] = \frac{1}{2}V_4, \end{aligned} \qquad (2.7)$$

and

$$[V_6, V_7] = 2V_8, \quad [V_7, V_8] = 2V_6, \quad [V_8, V_6] = 2V_7\,. \qquad (2.8)$$

The vector fields $V_1, ..., V_8$ are the generators of the infinitesimal symmetries transformations of Eqs. (1.1). These are the coordinate translations and the Galilean boost (V_4), which are common to both cases I and II. Moreover, for $\alpha = \beta$, Eqs. (1.1) admit the additional symmetry SU(2, C), which is expressed by the generators V_6, V_7, and V_8 satisfying the commutation rules (2.8) [9].

B. Group transformations

By integrating the infinitesimal operators $V_1, ..., V_8$, we provide the group transformations that leave Eqs. (1.1) invariant. These are, respectively,

$$V_1: \ \tilde{t} = t, \ \ \tilde{x} = x + \lambda, \ \ \tilde{u} = u, \ \ \tilde{v} = v, \qquad (2.9a)$$

$$V_2: \ \tilde{t} = t + \lambda, \ \ \tilde{x} = x, \ \ \tilde{u} = u, \ \ \tilde{v} = v, \qquad (2.9b)$$

$$V_3: \ \tilde{x} = x, \ \ \tilde{t} = t, \ \ \tilde{u} = u\,e^{i\lambda}, \ \ \tilde{v} = v\,e^{i\lambda}, \qquad (2.9c)$$

$$\begin{aligned} V_4: \ &\tilde{x} = x, \ \ \tilde{t} = t + \lambda\tilde{x}, \ \ \tilde{u} = ue^{\frac{i}{2}(\lambda\tilde{t} - \frac{1}{2}\lambda^2\tilde{x})}, \\ &\tilde{v} = ve^{\frac{i}{2}(\lambda\tilde{t} - \frac{1}{2}\lambda^2\tilde{x})}, \end{aligned} \qquad (2.9d)$$

$$\begin{aligned} V_5: \ &\tilde{x} = e^{\lambda}x, \ \ \tilde{t} = e^{\lambda/2}t, \\ &\begin{pmatrix} \tilde{u} \\ i\tilde{v} \end{pmatrix} = \sqrt{\tfrac{x}{\tilde{x}}} \begin{pmatrix} \cos k(\tilde{x} - x) & \sin k(\tilde{x} - x) \\ -\sin k(\tilde{x} - x) & \cos k(\tilde{x} - x) \end{pmatrix} \begin{pmatrix} u \\ iv \end{pmatrix}, \end{aligned} \qquad (2.9e)$$

$$\begin{aligned} V_6: \ &\tilde{x} = x, \ \ \tilde{t} = t, \\ &\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = e^{i\lambda\sigma_1} \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned} \qquad (2.9f)$$

$$\begin{aligned} V_7: \ &\tilde{x} = x, \ \ \tilde{t} = t, \\ &\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = e^{i\lambda(\cos 2kx\ \sigma_2 - \sin 2kx\ \sigma_3)} \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned} \qquad (2.9g)$$

$$\begin{aligned} V_8: \ &\tilde{x} = x, \ \ \tilde{t} = t, \\ &\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = e^{i\lambda(\sin 2kx\ \sigma_2 + \cos 2kx\ \sigma_3)} \begin{pmatrix} u \\ v \end{pmatrix}, \end{aligned} \qquad (2.9h)$$

where λ and σ_1, σ_2, and σ_3 are the group parameter and the Pauli matrices, respectively. In deriving (2.9), we have used the initial conditions $\tilde{x}(\lambda)\,|_{\lambda=0} = x$, $\tilde{t}(\lambda)\,|_{\lambda=0} = t$, $\tilde{u}(\lambda)\,|_{\lambda=0} = u$, and $\tilde{v}(\lambda)\,|_{\lambda=0} = v$.

Equations (2.9a)–(2.9h) tell us that if $u = f(x,t)$, $v = g(x,t)$ is a solution of the system (1.1), so are

$$u^{(1)} = f(x - \lambda, t), \quad v^{(1)} = g(x - \lambda, t), \qquad (2.10a)$$

$$u^{(2)} = f(x, t-\lambda), \quad v^{(2)} = g(x, t-\lambda), \tag{2.10b}$$

$$u^{(3)} = e^{i\lambda} f(x,t), \quad v^{(3)} = e^{i\lambda} g(x,t), \tag{2.10c}$$

$$u^{(4)} = f(x, t-\lambda x) e^{\frac{i}{2}(\lambda t - \frac{1}{2}\lambda^2 x)},$$

$$v^{(4)} = g(x, t-\lambda x) e^{\frac{i}{2}(\lambda t - \frac{1}{2}\lambda^2 x)}, \tag{2.10d}$$

$$\begin{aligned} u^{(5)} &= e^{-\frac{\lambda}{2}} \{ f(e^{-\lambda}x, e^{-\frac{\lambda}{2}}t) \cos[kx(e^{\lambda}-1)] \\ &\quad + ig(e^{-\lambda}x, e^{-\frac{\lambda}{2}}t) \sin[kx(e^{\lambda}-1)] \}, \\ v^{(5)} &= e^{-\frac{\lambda}{2}} \{ g(e^{-\lambda}x, e^{-\frac{\lambda}{2}}t) \cos[kx(e^{\lambda}-1)] \\ &\quad + if(e^{-\lambda}x, e^{-\frac{\lambda}{2}}t) \sin[kx(e^{\lambda}-1)] \}, \end{aligned} \tag{2.10e}$$

$$u^{(6)} = f(x,t)\cos\lambda + ig(x,t)\sin\lambda,$$

$$v^{(6)} = if(x,t)\sin\lambda + g(x,t)\cos\lambda, \tag{2.10f}$$

$$\begin{aligned} u^{(7)} &= (\cos\lambda - i\sin\lambda\ \sin 2kx) f(x,t) \\ &\quad + (\sin\lambda\ \cos 2kx) g(x,t), \\ v^{(7)} &= (-\sin\lambda\ \cos 2kx) f(x,t) \\ &\quad + (\cos\lambda + i\sin\lambda\ \sin 2kx) g(x,t), \end{aligned} \tag{2.10g}$$

$$\begin{aligned} u^{(8)} &= (\cos\lambda + i\sin\lambda\ \cos 2kx) f(x,t) \\ &\quad + (\sin\lambda\ \sin 2kx) g(x,t), \\ v^{(8)} &= (-\sin\lambda\ \sin 2kx) f(x,t) \\ &\quad + (\cos\lambda - i\sin\lambda\ \cos 2kx) g(x,t). \end{aligned} \tag{2.10h}$$

III. EXACT SOLUTIONS

As we have already mentioned, the method of symmetry reduction of a partial differential equation amounts essentially to finding the invariants (symmetry variables) of a given subgroup of the symmetry group admitted by the equation under consideration. A basis set of invariants for the generators V_j can be obtained by solving Eq. (2.1). Alternatively, one can resort to a direct equivalent procedure by using the group transformations (2.9).

The invariants can be exploited to provide exact solutions to Eqs. (1.1). By way of example, in this section we shall deal with the invariants related to the symmetry operators (a) $V_0 = V_1 + V_2 + V_3$, (b) V_4, (c) $V_1 + V_4$, and (d) V_5.

1. Case (a)

A set of invariants related to V_0 is

$$y = \tilde{t} - \tilde{x} = t - x,$$

$$U = \tilde{u}e^{-i\tilde{x}} = ue^{-ix}, \quad W = \tilde{v}e^{-i\tilde{x}} = ve^{-ix},$$

$$U_1 = \tilde{u}e^{-i\tilde{t}} = ue^{-it}, \quad W_1 = \tilde{v}e^{-i\tilde{t}} = ve^{-it}. \tag{3.1}$$

Inserting, for instance,

$$u(x,t) = U(y)e^{ix}, \quad v(x,t) = W(y)e^{ix}, \tag{3.2}$$

into Eqs. (1.1) (for $\alpha = \beta$) gives the pair of (ordinary) reduced equations

$$iU' + U - U'' - kW - \alpha(|U|^2 + |W|^2)U = 0, \tag{3.3a}$$

$$iW' + W - W'' - kU - \alpha(|U|^2 + |W|^2)W = 0, \tag{3.3b}$$

where

$$U = U(y),\ W = W(y),\ U' = \frac{dU}{dy},\ \text{and}\ W' = \frac{dW}{dy}.$$

Now, let us look for solutions to Eqs. (3.3) of the type

$$U(y) = p(y)e^{i\gamma y}, \quad W = q(y)e^{i\delta y}, \tag{3.4}$$

where p, q are real functions of y and γ, δ are real constants. In doing so, Eqs. (3.3) yield

$$(2\gamma - 1)p' + kq\sin(\delta - \gamma)y = 0,$$

$$(2\delta - 1)q' - kp\sin(\delta - \gamma)y = 0, \tag{3.5a}$$

$$p'' + (\gamma - \gamma^2 - 1)p + kq\cos(\delta - \gamma)y + \alpha(p^2 + q^2)p = 0,$$

$$q'' + (\delta - \delta^2 - 1)q + kp\cos(\delta - \gamma)y + \alpha(p^2 + q^2)q = 0. \tag{3.5b}$$

$$\left(p' = \frac{dp}{dy}, \quad q' = \frac{dq}{dy}\right).$$

Equations (3.5) produce some interesting configurations of Eqs. (1.1), such as the soliton profile and solutions expressed in terms of the elliptic Jacobi function sn( ).

To this aim, let us choose $\gamma = \delta = \frac{1}{2}$. Then, by defining $z = p + iq = \rho e^{i\pi/4}$ $(\rho = |z|)$, we have

$$\rho'^2 = (\tfrac{3}{4} - k)\rho^2 - \frac{\alpha}{2}\rho^4 + c, \tag{3.6}$$

where $\rho' = \frac{d\rho}{dy}$ and c is a constant of integration. The soliton profile comes from (3.6) for $c = 0$, by taking $\alpha > 0$, $k < \frac{3}{4}$. Precisely

$$p = q = \frac{1}{\sqrt{2}}\rho = \sqrt{(\tfrac{3}{4} - k)\frac{1}{\alpha}}\ \text{sech}\left[\sqrt{\tfrac{3}{4} - k}\,(y - y_0)\right], \tag{3.7}$$

where y_0 is an arbitrary constant. With the help of (3.7), from (3.2) we obtain

$$u = v = e^{\frac{i}{2}(t+x)}p(t - x), \tag{3.8}$$

with $p(t - x)$ given by (3.7).

We notice that for $c = 0$, and $\alpha < 0$, $k > \frac{3}{4}$, Eq. (3.6) leads to the solution

$$p = q = \frac{\rho}{\sqrt{2}} = \sqrt{(\tfrac{3}{4} - k)\frac{1}{\alpha}}\ \sec\left[\sqrt{k - \tfrac{3}{4}}\ (y - y_0)\right]. \tag{3.9}$$

In this case, we loose the soliton character of the profile. (The onset of the soliton depends on the parameters α and k.)

Another interesting solution linked to the symmetry operator V_0 arises for $c = k - 3/4 - |\alpha|/2 > 0$ and $\alpha < 0,\ k > \frac{3}{4}$. Indeed, from Eqs. (3.6), (3.4), and (3.2), we have

$$u = v = \frac{1}{\sqrt{2}} e^{\frac{i}{2}(t-x)} \mathrm{sn}\left[\sqrt{c}(t - x), h\right], \tag{3.10}$$

where sn( ) is the sinus elliptic function of modulus $h = \sqrt{\frac{|\alpha|}{2c}}$.

2. Case (b)

By using the basis of invariants

$$\tilde{x} = x, \quad U(x) = \tilde{u}e^{-i\tilde{t}^2/4\tilde{x}} = ue^{-it^2/4x},$$

$$W(x) = \tilde{v}e^{-i\tilde{t}^2/4\tilde{x}} = ve^{-it^2/4x}, \tag{3.11}$$

associated with the vector field V_4, from Eqs. (1.1), we find the pair of reduced equations

$$i\left(U' + \frac{1}{2x}U\right) + kW + \left(\alpha|U|^2 + \beta|W|^2\right) U = 0, \tag{3.12a}$$

$$i\left(W' + \frac{1}{2x}W\right) + kU + \left(\alpha|W|^2 + \beta|U|^2\right) W = 0, \tag{3.12b}$$

where $U' = \frac{dU}{dx},\quad W' = \frac{dW}{dx}$. For $\alpha = \beta$ and assuming that k is real, Eqs. (3.12) can be solved in the following way. First, let us divide (3.12a) and (3.12b) by U and W, respectively. Second, let us subtract the resulting equation corresponding to (3.12a) from that corresponding to (3.12b). Then, we obtain

$$i\left(WU' - UW'\right) + k\left(W^2 - U^2\right) = 0. \tag{3.13}$$

By introducing now

$$U = \frac{1}{2}(A + B), \quad W = \frac{1}{2}(A - B) \tag{3.14}$$

into Eq. (3.13), we can determine B in terms of A, namely

$$B = Ae^{-2i(kx+\delta_0)}, \tag{3.15}$$

where δ_0 is a constant of integration. Hence, putting the quantities (3.14) into Eq. (3.12a) and taking account of (3.15), after some manipulations we arrive at the solutions

$$u = \frac{a}{2\sqrt{x}}\, e^{i\left[\frac{t^2}{4x} + \alpha(1+|\lambda|^2)\, a^2 \ln x + \delta_0\right]} \times \left(e^{ikx} + \lambda\, e^{-ikx}\right), \tag{3.16a}$$

$$v = \frac{a}{2\sqrt{x}}\, e^{i\left[\frac{t^2}{4x} + \alpha(1+|\lambda|^2)\, a^2 \ln x + \delta_0\right]} \times \left(e^{ikx} - \lambda\, e^{-ikx}\right), \tag{3.16b}$$

where a is a real constant, and λ is a parameter defined by

$$\lambda = \frac{U_0 - V_0}{U_0 + V_0}, \tag{3.17}$$

with $U_0 = U(x,t)|_{x=x_0,t=t_0},\quad V_0 = V(x,t)|_{x=x_0,t=t_0}$.

Interesting special cases arise from (3.16) for $\lambda = 0$ and $\lambda = 1$. For $\lambda = 0$, Eqs. (3.16) yield

$$u = v = \frac{a}{2\sqrt{x}}\, e^{i\left(\frac{t^2}{4x} + kx + \alpha\, a^2 \ln x + \delta_0\right)}, \tag{3.18}$$

while when $\lambda = 1$, we obtain

$$u = \frac{a}{\sqrt{x}}\, e^{i\left(\frac{t^2}{4x} + 2\alpha\, a^2 \ln x + \delta_0\right)} \cos kx, \tag{3.19a}$$

$$v = i\frac{a}{\sqrt{x}}\, e^{i\left(\frac{t^2}{4x} + 2\alpha\, a^2 \ln x + \delta_0\right)} \sin kx. \tag{3.19b}$$

We notice that the mass density $|u|^2 + |v|^2$ is of the Coulomb type in the x variable and is time independent. A discussion of a possible physical interpretation of the class of solutions (3.18) is presented in Sec. IV.

Another class of exact solutions related to the generator V_4 can be found for $\alpha \neq \beta$ and $k = ik_0$, where k_0 is a real number.

In this case, let us look for solutions to Eqs. (3.12) of the type

$$U = \rho\, e^{i\theta}, \quad W = \rho\, e^{i\gamma}, \tag{3.20}$$

where ρ, θ, and γ are real functions of x. Inserting (3.20) into Eqs. (3.12) yields

$$\frac{\rho'}{\rho} + \frac{1}{2x} + k_0 \cos(\gamma - \theta) = 0, \tag{3.21a}$$

$$\theta' + k_0 \sin(\gamma - \theta) - (\alpha + \beta)\,\rho^2 = 0, \tag{3.21b}$$

$$\gamma' - k_0 \sin(\gamma - \theta) - (\alpha + \beta)\,\rho^2 = 0. \tag{3.21c}$$

By integrating this system and using (3.11), we give the following pair of exact solutions to Eqs. (1.1):

$$u = \rho\, e^{i\left(\frac{t^2}{4x} + \theta\right)}, \quad v = u\, e^{i \arcsin[\mathrm{sech} 2k_0(x_0 - x)]}, \tag{3.22}$$

where

$$\rho = \sqrt{\frac{\cosh 2k_0(x_0 - x)}{x}},$$

$$\theta = \frac{1}{2}\arcsin\left[\cosh 2k_0(x_0 - x)\right] + (\alpha + \beta)\, c^2 \int_{x_1}^{x} \frac{\cosh 2k_0(x_0 - x')}{x'}\, dx', \tag{3.23}$$

and c, x_0, and x_1, are real constants.

At this point, let us deal with the special choice $\theta = \gamma$. Then, Eqs. (3.21) lead to the solution

$$u = v = c\,\frac{e^{-k_0 x}}{\sqrt{x}}\, e^{i\frac{t^2}{4x}}\, e^{i\,(\alpha+\beta)\,c^2 \int_{x_1}^{x} \frac{e^{-2k_0 x'}}{x'}\,dx'}\,, \qquad (3.24)$$

where c and x_1 are constants. (For $x_1 \to \infty$, the integral at the right-hand side of (3.24) becomes $-E_1(2kx)$, where $E_1(\,)$ is the exponential integral function [10].) The "mass density" corresponding to the solutions (3.22) and (3.24) is

$$|u|^2 + |v|^2 = 2c^2\,\frac{\cosh 2k_0(x_0 - x)}{x},$$
$$|u|^2 = |v|^2 = c^2\,\frac{e^{-2k_0 x}}{x}. \qquad (3.25)$$

We point out that for $k = 0$, the mass densities (3.22) behave as $1/x$. Consequently, the presence of the parameter $k = ik_0$ induces a change of the Coulomb-like mass density.

3. Case (c)

A basis of invariants related to the symmetry operator $V_1 + V_4$ is given by

$$\eta = \tilde{t} - \frac{1}{2}\tilde{x}^2 = t - \frac{1}{2}x^2,$$

$$U(\eta) = \tilde{u}\,e^{-\frac{i}{2}\tilde{x}(\eta+\frac{1}{6}\tilde{x}^2)} = u\,e^{-\frac{i}{2}x(\eta+\frac{1}{6}x^2)},$$

$$W(\eta) = \tilde{v}\,e^{-\frac{i}{2}\tilde{x}(\eta+\frac{1}{6}\tilde{x}^2)} = v\,e^{-\frac{i}{2}x(\eta+\frac{1}{6}x^2)}. \qquad (3.26)$$

By assuming $\alpha = \beta$, and setting (3.26) into Eqs. (1.1), we get the reduced system

$$U'' - \frac{1}{2}\eta U + kW + \alpha\,(|U|^2 + |W|^2)\,U = 0, \qquad (3.27a)$$

$$W'' - \frac{1}{2}\eta W + kU + \alpha\,(|W|^2 + |U|^2)\,W = 0, \qquad (3.27b)$$

where $U' = \frac{dU}{d\eta}$. By requiring that U and W are real functions and $U = W$, Eqs. (3.24) lead to a special case of the second Painlevé equation [11], i.e.,

$$\frac{d^2\psi}{dz^2} = z\psi + 2\psi^3, \qquad (3.28)$$

where $z = 2^{\frac{2}{3}}\,(\frac{1}{2}\eta - k)$ and $\psi = 2^{\frac{1}{3}}\,\sqrt{-\alpha}\,U$.

4. Case (d)

A set of invariants arising from the generator V_5 is

$$\xi = \frac{\tilde{x}}{\tilde{t}^2} = \frac{x}{t^2},$$

$$I = \sqrt{\tilde{x}}(\tilde{u}\sin k\tilde{x} + i\tilde{v}\cos k\tilde{x}) = \sqrt{x}(u\sin kx + iv\cos kx),$$
$$J = \sqrt{\tilde{x}}(\tilde{u}\cos k\tilde{x} - i\tilde{v}\sin k\tilde{x}) = \sqrt{x}(u\cos kx - iv\sin kx), \qquad (3.29)$$

from which

$$K = \tilde{x}(|\tilde{u}|^2 + |\tilde{v}|^2) = x(|u|^2 + |v|^2).$$

Equations (3.29) imply

$$u = \frac{1}{\sqrt{x}}\,(I\sin kx + J\cos kx)\,, \qquad (3.30a)$$

$$v = \frac{i}{\sqrt{x}}\,(-I\cos kx + J\sin kx)\,. \qquad (3.30b)$$

Then, substitution from (3.30) into Eqs. (1.1) (for $\alpha = \beta$) yields

$$\frac{1}{2i}\,(I - 2\xi I') + 6\xi^2 I' + 4\xi^3 I'' + \alpha\left(|I|^2 + |J|^2\right) I = 0, \qquad (3.31a)$$

$$\frac{1}{2i}\,(J - 2\xi J') + 6\xi^2 J' + 4\xi^3 J'' + \alpha\left(|I|^2 + |J|^2\right) J = 0, \qquad (3.31b)$$

where $I = I(\xi)$, $J = J(\xi)$, $I' = \frac{dI}{d\xi}$, and $J' = \frac{dJ}{d\xi}$. A simple solution to the system (3.31) can be found supposing that $I(\xi)$ and $J(\xi)$ are real functions. In such a case, Eqs. (3.28) can be easily integrated. We have

$$u = \sqrt{\frac{-2}{\alpha}}\,\frac{\sin(kx + \phi)}{t},$$

$$v = -i\sqrt{\frac{-2}{\alpha}}\,\frac{\cos(kx + \phi)}{t}, \qquad (3.32)$$

from (3.29), where ϕ is a constant and α may take both positive and negative values.

IV. CONCLUSIONS

We have found some exact solutions to the system of equations (1.1) describing the propagation of waves in birefringent optical fibers. These equations have been analyzed in the framework of the Lie group theory. We have determined the associated symmetry algebra and the corresponding group transformations that leave Eqs. (1.1) invariant. Explicit configurations have been obtained both in the integrable and in the nonintegrable case. The subgroups (of the symmetry group) responsible for the soliton profile and for other interesting configurations have been provided. We point out that, for $\alpha \neq \beta$, the symmetry algebra is independent from the parameter k. This fact allows us to find exact solutions to the system (1.1) also in the case in which k is an imaginary parameter. In this situation, at least when $u = v$, k can be considered as the loss coefficient of the fiber. Then, the exact solution (3.24) is obtained. This is noteworthy in the sense that the related equation is not integrable.

Concerning a possible physical interpretation of the solutions (3.18) ($\alpha = \beta$, the integrable case), (3.22), and (3.24) ($\alpha \neq \beta$, the nonintegrable case), first we notice that in dealing with the invariance of a differential equation under a group of point transformations, together with the differential equation also the boundary and/or initial conditions must be invariant under the group. This reduces the order of the symmetry group allowed by the equation. In other words, this fact restricts the solutions to that class that is compatible with the boundary and/or initial conditions [12]. In our paper, we have limited ourselves to find the *widest* symmetry group relative to the system (1.1). We have not solved, say, a Cauchy problem, in the sense that we have not studied how a given initial condition evolves. In the framework of the Lie group theory, we have only solved *exactly* the couple of Eqs. (1.1). Therefore, a possible physical interpretation of our singular solutions should be necessarily indicative. A physical role of our "unusual" configurations can be argued as follows. A common feature of these configurations is that the corresponding "mass density" is time independent and becomes singular at $x = 0$. A first indication that these solutions may play a physical role is the fact that their mass density mimics a Coulomb or a Yukawa potential [see (3.25)]. Another aspect enforcing a possible physical interpretation is connected with the following considerations. For simplicity, let us take the case $k = 0$ and $u = v$ [the nonlinear Schrödinger (NLS) equation]. Then, it is well known that the solution of the NLS equation for a nonsoliton initial condition evolves into a soliton pulse and a decaying radiative part [13]. The asymptotic behavior for large distances is governed by the decaying radiative part. If

$$\psi_r = A_r e^{i\theta_r} \tag{4.1}$$

is the radiative part, where A_r, θ_r are real quantities, one has that mass (energy) conservation requires the following form for A_r [14]:

$$A_r = \frac{f(\frac{t}{x})}{\sqrt{x}}. \tag{4.2}$$

A natural choice for A_r is

$$A_r = \frac{A}{\sqrt{mx}} \operatorname{sech} \frac{t}{mx}, \tag{4.3}$$

where A and m are parameters to be determined. Thus, for the asymptotic form of the decaying radiative part, to be used in the mass (energy) invariants, one finds

$$\psi_r(x,t) = \frac{A}{\sqrt{mx}} \operatorname{sech} \frac{t}{mx} e^{\frac{it^2}{2x}}. \tag{4.4}$$

Keeping in mind the previous discussion, in our case we have that the solution (3.18) corresponds to a mass density given by the Coulomb-like expression

$$|u|^2 + |v|^2 = \frac{a^2}{4x}, \tag{4.5}$$

which is time independent.

Therefore, we shall interpret such a solution as a kind of *static* decaying radiative configuration. In other words, in our situation, we do not need to make convergent integrals (mass and energy invariants) in the time variable for the simple reason that we have a static mass density. The solutions (3.18) ($\alpha = \beta$) and (3.24) ($\alpha \neq \beta$) can be interpreted in the same manner. We point out that in all the above-mentioned configurations, the phase has not a pure Gaussian form, $\frac{t^2}{4x}$. This result is not surprising, because our configurations are *exact*, while, on the contrary, the phase involved in (4.4) is a consequence of a certain asymptotic expansion.

To conclude, we remark that the symmetry V_4 enables us to determine exact configurations, which have the form of decaying radiative signals. These signals may exist independently of the generation of the soliton profile. Therefore, they possess an own identity, in the sense that they have not to be regarded necessarily as "perturbations" concomitant with the onset of the soliton signal.

Finally, we think that the knowledge of exact solutions to Eqs. (1.1), which may be used with benefit as a guide for the development of perturbative techniques or for numerical calculations, could be a challenge for trying to create different experimental patterns.

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